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Pricing Behaviors on Networks: Some General Results for Two-way Networks ⋆

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Abstract. In this paper, we study monopolistic pricing behaviors within a two-way network. In this symbiotic production system, independent decision centers carry out an activity which concurs to the production of different system goods. The players are assumed to know the whole network. Due to this rationality, they try to capture a share of the profit of the firms who sell the system goods to the consumers. These double marginalization behaviors are studied within very general networks. Conditions with ensure existence and uniqueness are discussed. We even provided a complete characterization of an equilibrium. Potential applications are also discussed

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1. Introduction

Network activities play a crucial role in our economies. For instance, it is quite difficult to imagine what would happen if the transportation or the communication networks break down. These structures even cover a larger scope of economic activities than one presumes. In fact, if a network is defined by a set of activities with the property that (i) they are provided by independent decision centers, (ii) they are related by strong complementaries and (iii) they lead to the production of a good or a service, many commodities can be identified to network or system goods. A production process which requires subcontracting can, for instance, be viewed as a network. This general definition has two major implications. If these activities are controlled by independent players, each of these agents has a 'local' market power. Moreover if these activities are strongly related, one can expect that these independent players are aware of these relations: they have a network knowledge. If one now merges these two implications, one must concede that a player who acts as an intermediary knows that the firm who sells the goods to the consumers acts

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as a monopolist. This agent has therefore a strong incentive to capture a share of profit of the final seller.

In this paper, we try to illustrate this idea by studying the price formation of both intermediate and final goods. From that point of view, we take, contrary to Hendricks, Piccione and Tan [9, 10], the network as given. This one is described by a set of oriented edges which are identified to basic activities and which are chained together to obtain products sold to customers. One could for instance think at an airplane ticket from an origin to a destination which involves at least two different companies. We even restrict ourselves to a peculiar class of networks. As in usual transportation networks, we only consider 'two-way' networks[1] (see Economides–White [8] or Economides [7]). But as noted by Carter and Wright [3], if one introduces local monopoly power in a network in which, by the ('two-way' assumption, every final producer also sells intermediate goods, one deals with a 'symbiotic' production system which encompasses a large set of commodities or services including telecommunication, postal services or even the international flower delivering service called Interflora.

Our main objective is to study the pricing behaviors on such networks in a general setting. In a world in which each firm has constant average cost, we try to keep the network and the demands for transportation activities as general as possible in order to verify in a first step the existence of equilibrium prices. In a second step, we even obtain, under some additional assumptions, a complete characterization of these prices. This work can therefore by viewed as a preliminary but important step of a work in which one manipulates the data of this world (i.e. the unit costs, the network or the demands). Our main purpose is therefore to construct some 'reduced form' of a pricing game on a network, that is a form which is often taken as given like in the paper by Hendricks, Piccione and Tan [10]. We list, in Section 7 some potential applications of our result and develop an example of reciprocal access pricing in Internet. These applications largely rely on the manipulation of the basic data of a pricing game in a cooperative or a non-cooperative way.

The behaviors at work in this model have in some sense a double-marginalization flavor because each decision center has a network rationality and tries to capture a part of the profit of the agent who sells the goods to the consumers. However, in 'two-way' networks, each final seller also sells intermediate goods. A sequential approach, like the one used in the pancaking problem introduced by Laffont–Tirole [13, p. 185] does not fit. This is why we solves the game by a standard Nash equilibrium concept in which each player simultaneously chooses his margins

The model proposed in this paper works as follow. We consider a set of activities which are linked together in order to obtain travels from some origin to some destination. Several travels may of course exist between a given origin-destination couple. Each firm at the origin of a travel takes two decisions. It affects the demand to the different travels and charges a price to the costumer by taking a margin over its production cost. But this cost covers not only the cost of his own activity but

[1] An application to 'one-way' network can be found in Soubeyran and Stahn [18].

also the amount of money charged by the other decision centers which intervene in the production of the same composite good. From that point of view, we consider a game in which each final seller affects, in a first step, the customer to the different roads and in which, in a second step, each firm plays a Nash equilibrium in margins. The subgame perfect equilibria of this game are studied and general results are given on existence and uniqueness. We even give a full characterization of the set of solutions under quite reasonable assumptions. One of the most interesting result (see Proposition 7) is that these firms, for a given service, compete for equal profit shares independently of their unit cost level.

The paper will be organized as follows. The general setting is defined in sections 2 and 3. In fact Section 2 is devoted to the characterization of the physical network and to the set of commodities while Section 3 presents the behavior of different agents and the underlying game. In section 4, we present an example which illustrates the class of problem studied in this paper and presents our results. In Section 5, we analyze the subgame in which the agents choose margins, existence is proved under rather general assumptions and uniqueness is discussed. Section 6 is devoted to the optimal road choice and to the general solution of the game. In Section 7, we apply this approach to peering behavior on Internet and we describe some other applications. Finally, Section 8 is devoted to some concluding remarks. Proofs are relegated to an appendix.

2. The Market Structure

If one considers network goods, any description of a market structure must include on the one hand a physical description of the production network and on the other hand a definition of the set of services provided by this network. For that purpose we first characterize a ‘two-way’ transportation network (see Economide–White [8] or Economides [7]). This allows us, in a second step, to construct the set of commodities which are traded inside this ‘symbiotic’ network (see Carter and Wright [3] or Cricelli, Gastaldi and Levialdi [5]) and those whidh are sold to external customers.

2.1. THE NETWORK

A description of a network usually starts with a set of vertices which are identified to located decision centers and is followed by the definition of a set of edges which are viewed as connections or interactions between these decision centers. In this paper, we *do not* follow this standard approach. In fact, our approach of a network basically relies on a set of activities which are realized by independent decision centers and which are combined in order to create chains of activities. Each of these chains concurs to the production of a good or a service. This is why we start the definition of a network with a set A of activities $a \in A$ which are identified to oriented edges and are combined in order to obtain a travel or a path $t = (a_1^t, \dots, a_n^t)$

which goes through different activities. We denote by T the set of all available travels. The maps $o : T \to A$ and $e : T \to A$ associates to each travel t its first activity $o(t)$ and its last activity $e(t)$. The map $n : T \to \mathbb{N}$ returns the number $n(t)$ of activities in t and in order to make sure that the network structure makes sense we assume that $\forall t \in T, n(t) \geqslant 2$. We also introduce $c : T \to \mathbb{R}$, the cost $c(t)$ of one unit of good or service produced in the oriented chain $t \in T$. We even assume that the unit cost c_a^t of each activity along a travel can be clearly identified and, of course, that $c(t) = \sum_{a \in t} c_a^t$. If one associates a decision center to each basic activity, it also becomes important to identify the set $T_a = \{t \in T | a \in t\}$ of travels which contain $a \in A$ and to introduce two subsets $T_a^o = \{t \in T | a = o(t)\}$, and $T_a^i = T_a \backslash T_a^o$ which respectively describe the set of paths in which activity a is the first or an intermediate activity.[2]

This general setup of a production network fits with the description of a production process of a lot of composite (or system) goods. (see Matutes–Regibeau [15]). But if one is interested in transportation networks, one can introduce more structure. We first assume that a travel cannot contain a cycle or, in other words $\forall a, a' \in t, a \neq a'$. As a consequence T_a^o and T_a^i form a partition of T_a. One also notices that transportation networks are typically ‘two-way’ networks (see Economides–White [8] or Economides [7]). This means that if travel $t \in T$ then the reverse travel t^- is also available. These two travels are nevertheless considered as two different products even if one assumes that $c(t) = c(t^-)$, i.e. the cost of a travel is independent of the direction. Finally, we also assume that the cost of an elementary activity is independent of the travel, i.e. that $c_a^t = c_a$ for all $t \in T_a$. We however restrict these costs in a way to make sure that a longer travel in the sense that it requires more elementary activities is more costly, or in other words that if $n(t) > n(t')$ then $c(t) > c(t')$.

2.2. COMMODITIES, PRICES AND DEMANDS

By taking this network as given, let us now move to the description of the available transportation services. If one takes the point of view of a customer, one notices that he is not really interested in a travel itself. What is important for him is the existence of a connection between an origin and a destination. The service associated to a travel is therefore given by the map $S : T \to A \times A$ which associates to each $t \in T$ the origin destination couple $s(t) = (o(t), e(t))$. Moreover the set of available services is given by $S(T)$ an element of which is denoted by s. There also exists a demand D_s for each service $s \in S(T)$. Moreover, as Hendrick, Piccione and Tan [9], we assume that this demand is given by $D_s(p_s)$. In other words, we suppose that the demand for each origin-destination pair is independent of the price of the other destinations available at the same origin. The idea is that the customers

[2] At that point, it seems strange to treat the last activity as an intermediate one. But this point will become clear in the next subsection.

located at an origin and who wish to travel at a destination have no desire to travel anywhere else.

The reader however notices that several travels may induce the same service. If $\#A$ denotes the number of elements in A, this happens when $\#S^{-1}(s) = \#\{t \in T | (o(t), e(t)) = s\} > 1$.

In this case we assume that the firm which provides the activity at the origin of the travel has the opportunity to choose the proportion $(\alpha_s^t)_{t \in S^{-1}(s)}$ of customers affected to each travel. At this point one may wonder why the customers only care about the destination of the travel and not about the number of intermediate activities. This assumption is not crucial if one considers telecommunication networks but becomes important if one deals, for instance, with air carriers. We however decide to neglect this fact for basically two reasons. First, it happens that at equilibrium only the shortest roads are selected. One can, of course, always argue that this is not an argument. But remember that there is no congestion in our network. It this case, if consumers care about the length of the travel they surely choose as the firms the shortest one. From that point of view, time becomes important only if there is congestion. In this case a firm may have an incentive to redirect customers on longer travel at, of course, a cheaper price.

Finally, one also remarks that each service $s \in S(T)$ which requires a travel t with the property that $n(t) > 1$ is obtained by a sequence of basic transportation activities. The firm which sells the travel to the consumers must therefore purchase some intermediate services by the other firms along the travel t. Let[3] $(p_a^t)_{a \in t \setminus o(t)}$ denotes the unit prices charged by the intermediaries on travel t. In this case, one notices that the price charged by a intermediary for a same activity may change with the travel one considers. This follows from the fact that we assume that every agent has the knowledge of the whole network T and has the opportunity to discriminate the agents located at origin of the different travels.

If one takes for granted these definitions of the final and of the intermediate goods, one immediately notices that our network structure applies to railway, road or air carriers. But, following Carter and Wright [3] or Cricelli, Gastaldi and Levialdi [5], one even remarks that the production of travels can be identified to a symbiotic production system because this one has the following characteristics:

- each producer has a monopoly power in its own market and knows the network.
- each producer sells intermediate goods to an other transporter because we consider a 'two-way' network. Moreover, a producer which has access to a demand, produces both intermediate and final goods
- each producer must purchase the intermediate goods from an other producer

From that point of view, our approach also covers telephone services and other forms of telecommunication services like telex, telegram and even internet.

[3] By $t \setminus o$ we mean all the basic activities which compose travel t except the first one.

3. The Behaviors and the Underlying Game

Because each activity is provided by an independent decision center, we identify each component of A to a strategic agent. These firms have the ability to set the prices of the final or the intermediate services. However the important assumption we made is that these strategic agents have a network rationality. This means that they know the description of the network and, as a consequence, know exactly to which firm they sell an intermediate service and that this firm acts as a monopolist on the final market. An intermediate producer has therefore a strong incentive to charge a price which is higher than his marginal cost in order to capture a part of the profit of the final supplier. But in our setting, marginal costs are constant and equal to the average cost c_a. Hence choosing a price which is higher than this marginal cost is totally equivalent to choose the difference between this price and a given average cost, i.e. to take a margin $m_a^t = p_a^t - c_a$ over his unit cost. These margins, as the prices, are of course specific to each travel. Each agent has therefore the opportunity to set the following margins $(m_a^t)_{t\in T_a^i}$ where T_a^i is the set of all travels in which agent a intervenes. Moreover because one works with a symbiotic production system, one knows that $\forall a \in A$, $T_a^i \neq \emptyset$, because each firm always acts as an intermediary.

But some producers also have the opportunity to sell services to the consumers. In fact, by applying the service function S to the set T, each agent $a \in A$ can be characterized by a set $S_a = S(T_a^o)$ (which may be empty) of services provided to the customers. Let us now consider an agent for which $S_a \neq \emptyset$ and let us look at his behavior with respect to a service $s \in S_a$. This one has to take two decisions. He first has to affect the demand to the different travels $S^{-1}(s)$ which goes from the same origin to the same destination. In other words, he sets the proportions $(\alpha_s^t)_{t\in S^{-1}(s)}$ of the demand affected to each travel. But, as a monopolist, he also takes a margin over his unit production cost of service s. These margins are denoted by $(m_a^t)_{t\in T_a^o}$. However to choose these quantities, this agent must be able to compute the unit cost c_s of service s. With his knowledge of the network, he is conscious that on each travel $t \in S^{-1}(s)$ each intermediary charges a margin. The unit cost of travel $t \in S^{-1}(s)$ is therefore given by $c(t) + \sum_{a\in t\backslash o(t)} m_a^t$. The unit production cost of service s is obtained by using the proportions $(\alpha_s^t)_{t\in S^{-1}(s)}$ of customers affected to each travel. This one is given by $c_s = \sum_{t\in S^{-1}(s)} \alpha_s^t (c(t) + \sum_{a\in t\backslash o(t)} m_a^t)$ and can be decomposed into two terms: $r_s = \sum_{t\in S^{-1}(s)} \alpha_s^t \cdot c(t)$ which represents the real production cost of a unit of this network good and $M^t = \sum_{t\in S^{-1}(s)} \alpha_s^t (\sum_{a\in t\backslash o} m_a^t)$ which illustrates the profit capture. Having in mind that this firm also takes a margin, the price p_s of service s is given by:

$$p_s = \sum_{t\in S^{-1}(s)} \alpha_s^t \left(c(t) + \sum_{a\in t} m_a^t \right)$$

Moreover this decomposition of the price is assumed, under the network knowledge assumption, to be a common information to each player.

Let us now move to the computation of the profit function of agent $a \in A$. This one intervenes in the production process of each service $s \in S(T_a)$. For each $s \in S(T_a)$, he behaves either as a final seller or as an intermediary. In the first case the profit realized on service s is given by:

$$\pi_a^s = (p_s - c_s) \cdot D_s(p_s) = \left(p_s - \sum_{t \in S^{-1}(s)} \alpha_s^t \left(c(t) + \sum_{a \in t \backslash o} m_a^t\right)\right) D_s(p_s)$$
$$= \sum_{t \in S^{-1}(s)} \alpha_s^t \cdot m_a^t \cdot D_s \left(\sum_{t \in S^{-1}(s)} \alpha_s^t \left(c(t) + \sum_{a \in t} m_a^t\right)\right)$$

In the second case, one first needs to identify the subset of travels on which a intervenes for a given service. This set is given by $S^{-1}(s) \cap T_a$, the intersection of the set of activities which are required by service s and the set of travel in which a intervenes. Moreover for each of these travels he transports a proportion α_s^t of the demand and takes a margin over his unit cost. In this case, his profit is:

$$\pi_a^s = \sum_{t \in S^{-1}(s) \cap T_a} (p_a^t - c_a^t) \cdot \alpha_s^t \cdot D_s(p_s)$$
$$= \sum_{t \in S^{-1}(s) \cap T_a} \alpha_s^t \cdot m_a^t \cdot D_s \left(\sum_{t \in S^{-1}(s)} \alpha_s^t \left(c(t) + \sum_{a \in t} m_a^t\right)\right)$$

It is also a matter of fact to verify that if agent a sells service s then $S^{-1}(s) \cap T_a = S^{-1}(s)$, the profit of an intermediary or a final seller can therefore be describe by the same notation. If one now sums over all services in which a intervenes, his profit function is given by:

$$\pi_a(M_a, M_{-a}, \delta_a) = \sum_{s \in S(T_a)} \sum_{t \in S^{-1}(s) \in T_a} \alpha_a^t \cdot D_s \left(\sum_{t \in S^{-1}(s)} \alpha_s^t \left(c(t) + \sum_{a \in t} m_a^t\right)\right)$$

with

$$M_a = ((m_a^t)_{t \in T_a}),\ M_{-a} = \left(\left((m_{a'}^t)_{\substack{a' \in t \\ a' \neq a}}\right)_{t \in S^{-1}(s)}\right)_{s \in S(T_a)}$$

and

$$\delta_a = ((\alpha_s^t)_{t \in S^{-1}(s) \cap T_a})_{s \in S(T_a)}$$

In order to close the informal presentation of this game, it remains to discuss its timing. Concerning this point, we assume that the choice of the roads (i.e. the proportion of consumers affected to each travel) is taken before the choices of the different margins. However, in order to affect the consumers to the travels from an origin to a destination, we assume that the firms which deal with this choice are able to anticipate the different margins. This is why we use a notion of subgame perfect equilibrium[4] and begin to solve the game backwards. In other words, we allowed the final supplier to manipulate in a first step the production cost $r_s = \sum_{t \in S^{-1}(s)} \alpha_s^t c(t)$ of each service.[5]

If one wants to be more formal and defines the sub-game perfect equilibrium of this game, several notation are required. Let:

- $\#A$ denotes the number of elements of A, and Δ_+^k be the set $\Delta_+^k = \{x \in \mathbb{R}_+^k | \sum_{i=1}^k x_i = 1\}$,
- $\Delta_a = \prod_{s \in S(T_a)} \Delta^{\#(S^{-1}(s) \cap T_a)}$ a generic element of which is $\delta_a = ((\alpha_s^t)_{t \in S^{-1}(s) \cap T_a})_{s \in S(T_a)}$, $\Delta = \prod_{a \in A} \Delta_a$, and $\Delta_{-a} = \prod_{h \in A, h \neq a} \Delta_h$
- $\mathcal{M}_a = \mathbb{R}_+^{\#T_a}$ a generic element of which is $M_a = ((m_a^t)_{t \in T_\alpha})$ and let $\mathcal{M}_{-a} = \prod_{h \in A, h \neq a} \mathcal{M}_h$.

Because the agents observe the $\delta = (\delta_a)_{a \in A} \in \Delta$ before choosing their margins the strategy set of agent a is given by:

$$S_a = \left\{(\delta_a, M_a(\cdot)) \in \Delta_a \times \mathcal{M}_a^\Delta\right\}$$

and the payoff function of agent $a \in A$ is defined by

$$\begin{aligned} \pi_a : \mathcal{M}_a \times \mathcal{M}_{-a} \times \Delta_a &\to \mathbb{R} \\ (M_a, M_{-a}, \delta_a) &\mapsto \pi_a(M_a, M_{-a}, \delta_a) \end{aligned}$$

as given before. It follows that:

DEFINITION 1. A subgame perfect equilibrium (SPE) associated to the game $\Gamma = \langle S_a, \pi_a \rangle_{a \in A}$ is given by $(\delta_a^*, M_a^*(\cdot))_{a \in A} \in \prod_{a \in A} S_a$ with the property that

(i) $\forall \delta_a \in \Delta_a, \pi_a(M_a(\delta_a^*, \delta_{-a}^*), M_{-a}(\delta_a^*, \delta_{-a}^*), \delta_a^*) \geqslant \pi_a(M_a(\delta_a, \delta_{-a}^*), M_{-a}(\delta_a, \delta_{-a}^*), \delta_a)$

(ii) $\forall \delta \in \Delta, \forall M_a(\delta) \in \mathcal{M}_a, \pi_a(M_a^*(\delta), M_{-a}^*(\delta), \delta) \geqslant \pi_a(M_a(\delta), M_{-a}^*(\delta), \delta)$.

4. A Simple Example

In order to describe the main purpose of this paper, let us start with a simple example of a transportation network in which 6 independent companies are involved. They are denoted by $a \in A = \{1, \ldots, 6\}$ and realize a basic transportation activity at unit cost c_a. These companies are viewed (see Figure 1) as edges of a transportation graph G which is defined by a set T of travels which combines

[4] In fact, we do not really deal with a subgame perfect equilibrium because, as we will see later, the first stage of the game reduces to independent optimization problems.

[5] For another class of first step cost manipulation games see Long and Soubeyran [14].

these companies. The set S of transportation services offered to the customers is given by the couples origin-destination associated to each travel. A demand $D_s(p_s)$ is associated to each service. Under network rationality, price discrimination is allowed. Thus, each firm which is not at the origin of a travel has the opportunity to set prices for intermediate services associated to a given travel. A firm at the origin chooses the final price and routes the traffic. Moreover under constant unit costs, it is straightforward to verify that profit maximization is equivalent to set margins. One also notices that each final price of a service can be written in terms of the margins of the firms which contribute to this service. For instance, in our example (see Figure 1) the price $p_{(12)}$ can be written as:

$$\begin{aligned} p_{(12)} =& \alpha_{(152)} \cdot (m_1^{(152)} + m_5^{(152)} + m_3^{(152)} + c_1 + c_2 + c_3) \\ &+ \alpha_{(1462)} \cdot (m_1^{(1462)} + m_4^{(1462)} + m_6^{(1462)} + m_2^{(1462)} + c_1 + c_4 + c_6 + c_2) \end{aligned}$$

where $\alpha_{(152)}$ and $\alpha_{(1462)}$ denote the proportion of the demand $D_{(12)}(p_{(12)})$ effected to travel (152) and to travel (1462). The profit of agent a is therefore a function of the different margins and proportions. For instance (see Figure 1), the profit of agents 4 is given by:

$$\begin{aligned} \pi_4(m_4, m_{-4}, \alpha) =& \alpha_{(143)} \cdot m_4^{(143)} \cdot D_{(13)}(p_{(13)} + \alpha_{(341)} \cdot m_4^{(341)} \cdot D_{(31)}(p_{(31)}) \\ &+ \alpha_{(1462)} \cdot m_4^{(1462)} \cdot D_{(12)}(p_{(12)}) \\ &+ \alpha_{(2641)} \cdot m_4^{(2641)} \cdot D_{(21)}(p_{(21)}) \\ &+ \alpha_{(2543)} \cdot m_4^{(2543)} \cdot D_{(23)}(p_{(23)} \\ &+ \alpha_{(3452)} \cdot m_4^{(3452)} \cdot D_{(32)}(p_{(32)}) \end{aligned}$$

where $m_{-4} = (m_1, m_2, m_3, m_5, m_6)$ and $\alpha = (\alpha_1, a_2, \alpha_2)$. The whole pricing game can therefore be summarized in the following figure.

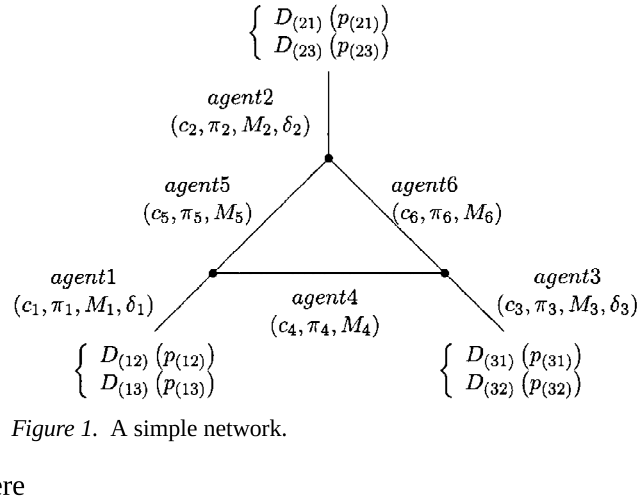

Figure 1. A simple network.

Where

$$T = \{(152), (251), (148), (341), (263), (362), (3651)(1462), (2641), (2543), (3452), (1563)\}$$
$$S = \{(12), (21), (13), (31), (23), (32)\}$$
$$\mathcal{M}_1 = \{M_1 = (m_1^{(152)}, (m_1^{(251)}, (m_1^{(143)}, (m_1^{(341)}, (m_1^{(3651)}, (m_1^{(1462)}, (m_1^{(2641)}, (m_1^{(1563)}) \in \mathbb{R}_+^8\}$$
$$\mathcal{M}_2 = \{M_2 = (m_2^{(152)}, (m_2^{(251)}, (m_2^{(263)}, (m_2^{(362)}, (m_2^{(2641)}, (m_2^{(1462)}, (m_2^{(2543)}, (m_2^{(3452)}) \in \mathbb{R}_+^8\}$$
$$\mathcal{M}_3 = \{M_3 = (m_3^{(143)}, (m_3^{(341)}, (m_3^{(263)}, (m_3^{(362)}, (m_3^{(3651)}, (m_3^{(2543)}, (m_3^{(3452)}, (m_3^{(1563)}) \in \mathbb{R}_+^8\}$$
$$\mathcal{M}_4 = \{M_4 = (m_4^{(143)}, (m_4^{(341)}, (m_4^{(1462)}, (m_4^{(2641)}, (m_4^{(2543)}, (m_4^{(3452)}) \in \mathbb{R}_+^6\}$$
$$\mathcal{M}_5 = \{M_5 = (m_5^{(152)}, (m_5^{(251)}, (m_5^{(3651)}, (m_5^{(2543)}, (m_5^{(3452)}, (m_5^{(1563)}) \in \mathbb{R}_+^6\}$$
$$\mathcal{M}_6 = \{M_6 = (m_6^{(263)}, (m_6^{(362)}, (m_6^{(3651)}, (m_6^{(1462)}, (m_6^{(2641)}, (m_6^{(1563)}) \in \mathbb{R}_+^6\}$$
$$\Delta_1 = \{\delta_1 = (\alpha_{(152)}, \alpha_{(1462)}, \alpha_{(143)}, \alpha_{(1563)}) \in \Delta_+^2 \times \Delta_+^2\}$$
$$\Delta_2 = \{\delta_2 = (\alpha_{(251)}, \alpha_{(2641)}, \alpha_{(263)}, \alpha_{(2543)}) \in \Delta_+^2 \times \Delta_+^2\}$$
$$\Delta_3 = \{\delta_3 = (\alpha_{(341)}, \alpha_{(3651)}, \alpha_{(362)}, \alpha_{(3452)}) \in \Delta_+^2 \times \Delta_+^2\}$$
$$\Delta_+^2 = \{x \in \mathbb{R}_+^2 | x_1 + x_2 = 1\}$$

Our paper proposes a general analysis of games of that type in which the proportions $\delta = (\delta_1, \delta_2, \delta_3)$ are set before the margins $M = (M_1, M_2, M_3, M_4, M_5, M_6)$. These margins must therefore be viewed as functions of the road choices. But in order to spare notations, we do not explicitely introduce this relation in the rest of the paper. The existence of a solution to this game only requires the existence of C^1 strictly decreasing demand functions with the property that these demands are zero after some price. The proof goes as follow. One first notices that the problem can be decomposed service by service; the game reduced to a service is called a generic game. Secondly, we remark that the equilibrium of a generic game can be obtained as a fixed point of a function which solves a parametrized optimization problem. We conclude in a third step to existence by using Milgrom–Shannon's [16] comparative static results and by applying Tarski's [20] fixed point theorem. Moreover if the demands are assumed to be C^2 and strictly log-concave, we provide a complete characterization of the solutions of games of that type. For instance, if in our example the demands are all given by $D(p_s) = 1 - p_s$ and the costs by $c = (.1, .2, .3, .2, .2, .1)$ then an equilibrium, by applying Proposition 9, is immediately given by:

$$\begin{cases} M_1 = M_2 = (.125, .125, .1, .1, 0, 0, 0, 0) \quad M_3 = (.1, .1, .1, .1, 0, 0, 0, 0) \\ M_4 = M_6 = (.1, .1, 0, 0, 0, 0) \quad M_5 = (.125, .125, 0, 0, 0, 0) \\ \delta_1 = \delta_2 = \delta_2 = (1, 0, 1, 0) \\ p_{(1,2)} = p_{(2,1)} = .875 \quad p_{(1,3)} = p_{3,1)} = p_{(3,2)} = p_{(2,3)} = .9 \\ \pi_1 = \pi_2 \simeq .05 \quad \pi_3 = .04 \quad \pi_4 = \pi_6 = .02 \quad \pi_5 \simeq .03 \end{cases}$$

Proposition 9 can therefore be viewed as the reduced form of this game. This opens a wide class of economic applications in which the data of this model (i.e.

the graph, the demands or the costs) can be manipulated by means of a cooperative or a non-cooperative game. Some applications are stressed in Section 8. The reader also notices that, for a given service, the firms compete for equal profit shares independently of their unit cost level. For instance for service (12) each firm obtains the same profit given by $(.125)^2$ but none of the them has the same unit cost. Let us now move to the general description of games of that type.

5. A Generalized Double Marginalization Game

At this stage of the study, we focus on point (ii) of the definition of an equilibrium. If fact, it is well knows that for sub-game perfect equilibria every deviation from the equilibrium path must be followed by Nash equilibrium strategies of the sub-game which is in continuation. But our game only consists in two steps. We therefore simply has to study the Nash equilibrium of a game in which the firms choose their margins whatever the road choices are. In the first subsection, we briefly recall the definition of this Nash equilibrium and verify that this game can be decomposed into a family of generic games. As a consequence, one simply needs to study one of these generic games in order to know the properties of the whole game. This is done in a second subsection.

5.1. THE EQUILIBRIUM: DEFINITION AND DECOMPOSITION

The choice of margins is motivated by two facts. On the one hand each firm as a monopolist has an incentive to exploit his market power. On the other hand, the knowledge of the network encourages each agent to capture a part of the profit of the other firms. This is why strategic interactions appear. From that point of view, an equilibrium coincides to a situation in which no agent has an incentive to change his margins, given the strategies of the other firms. In other words:

DEFINITION 2. An equilibrium is a vector of margins $(M_a^*)_{a\in A}$ with the property that:

$$\forall a \in A, \forall M_a \geqslant 0, \quad \pi_a(M_a^*, M_-^* a, \delta_a) \geqslant \pi_a(M_a, M_{-a}^*, \delta_a)$$

where the road choices $(\delta_a)_{a\in A}$ are taken as given.

If one comes back to the definition of the profit function, one notices that the optimization problem of a firm is separable with respect to the different services to which an agent contributes. One therefore expects that the capture game also satisfies this property. In other words, one hopes that this game can be decomposed with respect to each service $s \in S(T)$.

In order to make this point more precise, let us concentrate on one particular service s. It is a matter of fact to identify the agents who contribute to its production. They belong, by construction, to the set $I_s = \{a \in A | t \in T_a \text{ for some } t \in S^{-1}(s)\}$.

Moreover if one denotes by $\sigma_{i,s} = \sum_{t \in S^{-1}(s) \cap T_i} \alpha_s^t \cdot m_i^t$ the global profit share[6] which is taken by each of these agents $i \in I_S$ and by $r_s = \sum_{t \in S^{-1}(s)} \alpha_s^t \cdot c(t)$ the cost of service $s \in S$, one easily verifies that the equilibrium margins given by Definition 2, and which are associated to a specific service are also equilibrium margins of a game restricted to this given service. More formally, the following proposition is satisfied.

PROPOSITION 1. *Let $(M_a^*)_{a \in A}$ be an equilibrium of the whole game and let us restrict to the equilibrium margins $(((m_a^{*t})_{a \in t})_{t \in S^{-1}(s)})$ associated to service $s \in S(T)$. If one defines I_s by $I_s = \{a \in A | t \in T_a \text{ for some } t \in S^{-1}(s)\}$ by $r_s = \sum_{t \in S^{-1}(s)} \alpha_s^t \cdot c(t)$ and $\sigma_{i,s}$ by $\sigma_{i,s} = \sum_{t \in S^{-1}(s) \cap T_i} \alpha_s^t \cdot m_i^t$ for all $i \in I_s$, then one can assert that:*

$$\forall s \in S(T), \forall i \in I_s, \quad \sigma_{i,s}^* \in \arg\max_{\sigma_{i,s} \in \mathbb{R}_+} \sigma_{i,s} D_s \left(\sigma_{i,s} + \sum_{j \in I_s \setminus \{i\}} \sigma_{j,s}^* + r_s \right)$$

One can even go a step further. In fact, one also notices that the margins which contribute to the definition of the price p_s of a service $s \in S(T)$ never contribute to the definition of an other price because we have assumed that each firm is able to discriminate the final suppliers. From that point of view, it seems possible to study an equilibrium of a total game only by looking for an equilibrium service by service, or, in other word, by solving enough generic games defined in the following way:

DEFINITION 3. Let $D : \mathbb{R}_+ \to \mathbb{R}_+$ a C^1 function with the property that (i) $\forall p \geqslant \bar{p}$, $D(p) = 0$, (ii) $\forall p < \bar{p}$, $D(p) > 0$, (iii) $\forall p \in]0, \bar{p}[$, $D'(p) < 0$, and r be a strictly positive scalare such that $r < \bar{p}$. An equilibrium of a generic game is a vector $(\sigma_i^*) \in \mathbb{R}_+^{\#I}$ with the property that:

$$\forall i \in I, \quad \sigma_i^* \in \arg\max_{\sigma_i \in \mathbb{R}_+} \sigma_i D \left(\sigma_i + \sum_{j \in I \setminus \{i\}} \sigma_j^* + r \right)$$

It therefore remains to verify that the study of appropriate generic games induces an equilibrium of the whole game. This point is proved in the next proposition.

PROPOSITION 2. *Let $A_s = \{a \in A | t \in T_a \text{ for some } t \in S^{-1}(s)\}$, $r_s = \sum_{t \in S^{-1}(s)} \alpha_s^t \cdot c(t)$ and $D(\cdot) = D_s(\cdot)$ for each $s \in S(T)$. If one denotes by $(\tilde{\sigma}_a^s)_{a \in A_s}$ a solution of the s^{th} generic game then the margins $((\tilde{m}_a^t)_{a \in t})_{t \in T}$ given by:*

$$\forall s \in S(T), \forall a \in A_s, \sum_{t \in T_a \cap S^{-1}(s)} \alpha_s^t \cdot \tilde{m}_a^t = \sigma_a^s$$

describe an equilibrium of the complete game.

[6] It is a matter of fact to verify that $\sigma_{i,s}$ can be an indicator of the share of profit that takes agent i on service s because the real profit share is given by $\frac{(\sum_{t \in S^{-1}(s) \cap T_a} \alpha_s^t \cdot m_a^t) D_s(p_s)}{(p_s - r_s) D_s(p_s)} = \frac{\sigma_{i,s}}{\sum_{i \in I_S} \sigma_{i,s}}$.

These two propositions are very interesting. They tell us that if one wants to know some properties of total double marginalization game, one simply needs to study the generic game given in Definition 3. So let us concentrate on this equilibrium.

5.2. SOME PROPERTIES OF THE GENERIC GAME

If one looks at this class of games, one notices that some trivial equilibria always exist. For instance, any vector of strategies in the set $\{(\sigma_i)_{i\in I} \in \mathbb{R}^I_+ | \forall i \in I, \Sigma_{-i} = \sum_{j\in I\setminus\{i\}} \sigma_j \geqslant \bar{p} - r\}$ is a Nash equilibrium. In this case, any individual deviation does not affect the profits because the demand remains at 0.

REMARK 1. From a pure logical point of view, one can even assert that a SPE associated to the game $\Gamma = \langle S_a, \pi_a\rangle_{a\in A}$ exists if $\forall s \in S(T)$, $\exists \bar{p}_s > 0$ with the property that $\forall p_s \geqslant 0$, $D(p_s) = 0$.

The idea is the following. By Proposition 2, let us look at a given service s, and let us compute $\min_{(\alpha_s^t)_{t\in S^{-1}(s)}} r_s = \min_{t\in S^{-1}(s)}\{c(t)\}$. Because the $(\sigma_a^s)_{a\in A_S}$ has no upper bound, one can choose $\tilde{\sigma}_a^s$, such that $\forall a \in A_s$, $\sum_{j\in A_s\setminus\{a\}} \hat{\sigma}_a^s \geqslant \bar{p}_s - \min_{t\in S^{-1}(s)}\{c(t)\}$. By construction $(\tilde{\sigma}_a^s)_{a\in A_s}$ is a trivial Nash equilibrium for service s. But we can even claim that this vector is a trivial equilibrium whatever the road choices are because we have chosen r_s as small as possible. It remains to apply this argument to each service and, in the spirit of Proposition 3, to set $\forall a \in A_s$, $\forall s \in S(T)$, $\forall t \in T_a \cap S^{-1}(s)$, $\tilde{m}_a^t = \sigma_a^s$. The margins reconstructed in this way ensure the existence of trivial Nash equilibrium for each service whatever the road choices are. Hence any road choice $\delta^* \in \Delta$ associated to, $(M_a^*(\delta))_{a\in A} = ((\hat{m}_a^t)_{t\in T_a})_{a\in A}$ $\forall \delta \in \Delta$ is a SPE.

These trivial equilibria are however not very interesting from an economic point of view. So let us restrict to non-trivial equilibria that is to equilibria with the property that $\Sigma^* \equiv \sum_{i\in I} \sigma_i^* \leqslant \bar{p} - r \equiv \bar{\Sigma}$ and let us establish their existence. In order to obtain this result, one first notes that a change in the strategy of any player affects the demand in the same way. As a consequence, one can hope that symmetry follows. If one uses standard technics which were developed for Cournot models (see for instance Amir [1]), one can even expect that any symmetric equilibrium can be obtained by a computation of a very simple fixed point in $\mathbb{R}$ the function of which is, up to slight changes, a parametrized solution of an optimization problem. More precisely, one can show that:

PROPOSITION 3. *Let* $\pi : [0, \bar{\Sigma}] \times [0, \bar{\Sigma}] \to \mathbb{R}$ *be given by* $\pi(\Sigma, \Sigma_{-i}) = (\Sigma - \Sigma_{-i}) \cdot D(\Sigma + r)$.
Let $C : [0, \bar{\Sigma}] \to 2^{[0,\bar{\Sigma}]}$ *be defined by* $C(\Sigma_{-i}) = [\Sigma_{-i}, \bar{\Sigma}]$
Let $\phi(\Sigma_{-i}) = \{\Sigma \in \mathbb{R}_+ | \Sigma \in \arg\max_{\Sigma\in C(\Sigma_{-i})} \pi(\Sigma, \Sigma_{-i})\}$
Let $\varphi : [0, \bar{\Sigma}] \to 2^{[0,\bar{\Sigma}]}$ *such that* $\varphi(\Sigma_{-i}) = \frac{n-1}{n}\phi(\Sigma_{-i})$ *with* $n = \#I$

One verifies that Σ^*_{-i} *is the fixed-point of* φ *if and only if* $\sigma^*_i = \frac{1}{n-1}\Sigma^*_{-i}$ *is a symmetric Nash equilibrium of the generic game.*

It remains to verify that all non trivial equilibria of the generic game are symmetric. If this is the case this surely simplifies the existence proof. This result also has a strong economic content. It implies, if one remembers that the σ_i are profit share indicators, that the firms obtain the same profit share independently of their costs or in other words that they take the same margins and obtain the same profits. So let us establish that:

PROPOSITION 4. *Under the assumption that* $\forall p \in]0, \bar{p}[$ $D'(p)$ *is well-defined,* $D'(p) < 0$, *and* $\forall p \geqslant \bar{p}$ $D(p) = 0$, *every non trivial equilibrium of a generic game is symmetric.*

Because each non trivial equilibrium of the generic game is obtained by the computation of a fixed-point the function of which is a solution of a parametrized optimization problem. It therefore remains to characterize the set-valued map $\phi(\Sigma_{-i})$. By applying Milgrom–Shannon's comparative static [16] approach, one establishes that:

PROPOSITION 5. *Under the preceding assumptions, every selection* $f(\Sigma_{-i})$ *from the set-valued map* $\phi(\Sigma_{-i}) = \arg\max_{\Sigma \in C(\Sigma_{-i})} \pi(\Sigma, \Sigma_{-i})$ *is well-defined and non decreasing.*

In order to show existence, it remains to verify that $\varphi(\Sigma_{-i}) = \frac{n-1}{n}\phi(\Sigma_i)$ has a fixed-point. But any selection $g : [0, \bar{\Sigma}] \to [0, \bar{\Sigma}]$ of $\varphi(\Sigma_{-i})$ is, by the preceding proposition, well-defined and non decreasing. It follows from Tarsky's fixed-point theorem [20] that:

PROPOSITION 6. *If*

(i) $\exists \bar{p} > r, \forall p \geqslant \bar{p}, D(p) = 0$

(ii) $\forall p \in]0, \bar{p}[$, $D'(p)$ *is well-defined and* $D'(p) < 0$

the generic game induced by this demand function has at least one non-trivial symmetric equilibrium. Moreover every non-trivial equilibrium is symmetric.

As a corollary, one immediately notices that this proposition insures the existence of a non-trivial equilibrium in the more general game defined on our network.

REMARK 2. If each demand $D_s(p)$ satisfies the preceding assumption, then the game $\Gamma = \langle S_a, \pi_a \rangle_{a \in A}$ admits a SPE with the property that there is no trivial equilibrium in the second step of the game.

To verify this point, fix any vector of road choice $\delta^* \in \Delta$, define the required number of generic game (Proposition 1), by Proposition 6, choose for each of them a non-trivial equilibrium, and select equilibrium margins $(M^*_a)_{a \in A}$ by applying Proposition 2. Now let us construct, for each $a \in A$, the strategies $(\delta^*_a, M^*_a(\cdot))$ where $M^*_a(\cdot)$ is given by $M^*_a(\delta^*_a) = M^*_a$ and $\forall \delta \neq \delta^*_a$, $M^*_a(\delta) = (\hat{m}^t_a)_{t \in T_a}$ given in Remark 1. These strategies define a SPE. It is of course a Nash equilibrium: nobody has an incentive to deviate at step 2 because a Nash equilibrium is played and if one agent deviates at step 1, he is punish because the other react by playing a strategy which neutralizes his profit. But this strategy is even a Nash equilibrium of the subgame which starts after the deviation (see Remark 1). Hence we have an SPE.

This kind of equilibria are again not really satisfying from an economic point of view. They of course implement a non trivial equilibrium in the second step of the game, but the threat is so hard that the equilibrium path leaves the road choices completely indeterminate Moreover, we are even not able to compute in a simple way the second period Nash Equilibrium. This is why we decided to look for stronger restriction on the model in order to make sure that (i) M_a^* is unique (at least for travels which are activated) and easily computable (ii) the threat can be based on a non-trivial equilibrium. Let us first concentrate on the uniqueness issue.

Roughly speaking, this property is obtained if the solution $\phi(\Sigma_{-i})$ of the optimization problem is a continuous function the slope of which is smaller than In order to obtain this result, we assume that the demand is strictly log-concave. This assumption can seem very arbitrary. But if one introduces discrete choices of heterogeneous consumers, this assumption can be simply based on a log-concave distribution of the characteristics of the consumers (see Caplin–Nalebuff [2]). Under this restriction, one can prove the following proposition.

PROPOSITION 7. *If one also assumes that $D(p)$ is strictly log-concave, C^2 on $]0, \bar{p}[$ and that $\lim_{p\to\bar{p}'} D'(p) < 0$, then the non-trivial equilibrium of generic game is unique. Moreover if one denotes by $\pi_i(n, r)$ the equilibrium profit of agent i, one observes that $\forall i \in I$, $\pi(n, r) = \pi(n, r)$, that $\partial_n \pi(n, r) < 0$ and that $\partial_r \pi(n, r) < 0$.*

The reader surely notices that:

REMARK 3. For a generic service, each firm essentially compete for equal price share and profit independently of its unit cost. Moreover the profit are the same for each firm which contributes actively to a given service. One observes some profit equilisation rule.

This result is strongly related to the fact that the different activities are complementary. This basic property is fundamental in several applications because it crucially structures the nature of the equilibrium profits.

6. Toward a Characterization of the Complete Solution

The aim of this section is to obtain a complete characterization of the solution of the transportation game. But this requires that we solve in a first step the optimal road choice problem which is simply an optimization problem.

6.1. THE OPTIMAL ROAD CHOICE

In order to perform this task, one needs (i) to define the subset $A_r \subset A$ of agents who have this choice opportunity and (ii) to construct their profit functions. The first question is quite easy to solve. One considers the set $A_r = \{a \in A | T_a^o \neq \emptyset\}$ of agents which are at least at the origin of one travel. One however needs to be more careful concerning the computation of their profit function.

So let us start with an agent $a \in A_r$ and a service $s \in S(T_a^o)$. By Proposition 2, one knows that the equilibrium margins are obtained by solving a generic game the parameters of which are the number $\#I_s$ of agents concerned by the production of service s and the unit network production cost given by the scalare $r_s = \sum_{t \in S^{-1}(s)} \alpha_s^t \cdot c(t)$. By Proposition 7, one also knows that each agent $a \in I_s$ obtains the same profit level $\pi_s(\#I_s, r_s)$. It remains to sum over all services $s \in S(T_a^o))$ which are sold by agent $a \in A_r$ in order to obtain his profit function. By affecting the customers to the different travels each agent $a \in A_r$ therefore solves

$$\max_{((\alpha_s^t)_{t \in S^{-1}(s)})_{s \in S(T_a^o)}} \sum_{s \in S(T_a^o)} \pi_s(\#A_s, r_s) \quad \text{s.t. } \forall s \in S(T_a^o) \sum_{t \in S^{-1}(s)} \alpha_s^t = 1$$

In this case, an agent has the opportunity, for each service, to manipulate both $\#A_s$ and r_s, i.e. the number of activities which concurs to the production of service and the network production cost. As a consequence no game is involved and the road choice simply results from an optimization problem service by service. Moreover if one also assumes that the different demand functions are log-concave, one knows that the profit functions are decreasing with the number of agents who participate to the production of the service. Each seller $a \in A_r$ has therefore a strong incentive to only activate one travel: the shortest one because in this case a minimum of intermediaries are needed. Moreover if several shortest travels exist, he chooses the cheapest one. More formally, one can show that:

PROPOSITION 8. *If $t_s^* \in \arg\min_{t \in T_s^n}\{c(t)\}$ and $T_s^n = \{t \in T | t \in \arg\min_{t \in S^{-1}(s)}$ $\{n(t)\}\}$ then an optimal road choice for service $s \in S(T)$ is $\alpha_s^{t_s^*} = 1$ and $\alpha_s^t = 0$ for all $t \in S^{-1}(s)$ and $t \neq t_s^*$.*

6.2. A DESCRIPTION OF THE SOLUTIONS OF THE TRANSPORTATION PROBLEM

If one now puts together the results obtained in Proposition 7 and Proposition 8, one can give a complete characterization of an equilibrium of our game.

PROPOSITION 9. *If for $s \in S(T)$, $D_s(p_s)$ satisfies*

(i) $\exists \bar{p}_s > 0, \forall p_s \geqslant \bar{p}_s, D_s(p_s) = 0$
(ii) $\forall p_s \in]0, \bar{p}_s[$, $D_s(p_s)$ *is at least* C^2, $D_s'(p_s) < 0$ *and* $\lim_{p \to \bar{p}_s} D_s'(p_s) < 0$
(iii) $\forall p_s \in]0, \bar{p}_s[$, $D_s(p_s)$ *is strictly log-concave*
(iv) $\bar{p}_s > \max_{t \in S^{-1}(s)}\{c(t)\}$

and if $t_s^* \in \arg\min_{t \in T_s^n}\{c(t)\}$ *and* $T_s^n = \{t \in T | t \in \arg\min_{t \in S^{-1}(s)}\{n(t)\}\}$ *then an equilibrium in which every service* s *is available has the following properties for all* $s \in S(T)$*:*

(i) $\forall s \in S(T)$, $\alpha_s^{t_s^*} = 1$ *and* $\alpha_s^t = 0$ *for all* $t \in S^{-1}(s)$ *and* $t \neq t_s^*$.

(ii) $\forall s \in S(T)$, $m_s^{t_s^*} = m_s$ *and* $m_s^t = 0$ *for all* $t \in S^{-1}(s)$ *and* $t \neq t_s^*$.

(iii) $\forall s \in S(T)$, m_s *solves*

$$m_s = -\frac{D_s(n(t_s^*) \cdot m_s + c(t_s^*))}{D_s'(n(t_s^*) \cdot m_s + c(t_s^*))}$$

(iv) *the price levels are* $\forall s \in S(T)$, $p_s = n(t_s^*) \cdot m_s + c(t_s^*)$

(v) *the equilibrium profits are* $\forall a \in A$

$$\pi_a = \sum_{s \in S(T_a)} \mathbb{I}_{T_s^*} \frac{(D_s(p_s))^2}{-D_s'(p_s)} \quad \text{with } \mathbb{I}_{t_s^*} = \begin{cases} 1 & \text{if } t_s^* \in S^{-1}(s) \\ 0 & \text{else} \end{cases}$$

In other words, if one assumes that the demand functions (i) admit reservation prices, (ii) are strictly decreasing for strictly positive demands, (iii) are log-concave and that (iv) each reservation price $\bar{p}_s$ does not *per se* exclude travel t which realizes service s, our problem has a well-defined solution. In this case, each seller chooses the shortest and cheapest travel. The margins along a selected travel are the same across the firms and can be computed by means of a simple equation.

7. Some Uses of This Result

In this section, we quickly present an example of an application of our result which deals with ‘peering’ behaviors in Internet. We discuss, in a second step, some other extensions of this model. All of them rely on a manipulation of the basic data which defined the transportation game and lead to some interesting economic applications.

7.1. A SIMPLE ILLUSTRATION

If one considers Internet, one often distinguishes the Internet Service providers (IFS) and the Internet Backbone Providers (IPB) (see for instance Crémer–Rey–Tirole [4] or Dang Nguyen–Pénard [6]). The last ones transmit data over large regions of the world using long-haul fiber-optic cables and they often exchange data to each others under ‘peering agreements’. In other words they accept to route all traffic which comes from an other IBP and which is delivered to one of their customers without any charge. Our model provides a very simple explanation of this fact as long as one introduces an explicit negotiation step.

To keep the example as simple as possible, let us only introduce two interconnected IBPs which are identified in our terminology to basic activities. They are denoted by $A = \{a, b\}$ and their unit costs are given by c_a and c_b. Concerning the travels, one notices that a message can either stay in the same backbone or goes to a customer of the other one. The set T of travels is therefore given by $T = \{(a), (b), (a, b), (b, c)\}$. Concerning the services sold to the consumers, we however need to slightly adjust the reference model. In fact, on Internet the customers do not buy a travel, they look for an access. So let us denote by $D_i(p_i) i = a, b$ the demand to each provider and let us assume that a proportion α_i of the access demand $D_i(p_i)$ goes to customers of network $j \neq i$. Because each IBP acts as a monopolist, he takes a margin m_i over his costs. These unit costs cover his own production costs c_i and the unit price charged by the other IBF in order to allow a proportion α_i of the demand to access network j. If one denotes by m^i_j the margin charged by j over a travel which comes from i, the price p_i can be decomposed in the following way:

$$p_i = m_i + (1 - \alpha_i) \cdot c_i + \alpha_i \cdot (c_i + m^i_j + c_j) = m_i + \alpha_i \cdot m^i_j + \underbrace{(c_i + \alpha_i \cdot c_j)}_{r_i}$$

and the profit functions are given by

$$\pi_i(m_i, m^j_i, m_j, m^i_j) = m_i \cdot D_i(p_i) + \alpha_j \cdot m^j_i \cdot D_j(p_j)$$

From that point of view, one deals with a standard transportation game in which agent i chooses m_i and m^j_i. The solutions of this game are well-known under some restrictions on $D_i(p_i)$. Moreover, it is immediate by Propositions 7 and 9, that no peering behaviors appear because m_i solves for $i = a, b$

$$m_i = -\frac{D_i(2m_i + r_i)}{D'_i(2m_i + r_i)} \quad \text{and} \quad m^j_i = \frac{m_j}{\alpha_j}$$

This is not really surprising because peering behaviors rely on an agreement. This is why a notion of co-opetition is required (see Nalebuff–Brandenburger [17]). In other words, one has to consider a game in which the two IBPs negotiate in a first step their reciprocal access charges (i.e. the m^i_j) by maximizing their joint profit and, in a second step, choose independently their own margins. If one solves this problem backwards, one first has to consider a transportation game in which the m^i_j are taken as given, one then computes the optimal profit level $\hat{\pi}_i(m^b_a, m^a_b)$ as functions of m^b_a and m^a_b, and one finally chooses the reciprocal access charges by maximizing $\hat{\pi}_a(m^b_a, m^a_b) + \hat{\pi}_b(m^a_b, m^b_a)$ over m^b_a and m^a_b.

If one considers, in this example, linear demand functions $D_i(p_i) = d_i - c_i \cdot p_i$, it is a matter of fact to verify that the margins $m_i \; i = a, b$ which solve the transportation game are given by $m_i = \frac{1}{2c_i}(d_i - c_i(\alpha_i \cdot m^i_j + r_i))$ and that the profit functions are given by:

$$\hat{\pi}_i(m^b_a, m^a_b) = \frac{1}{4c_i}(d_i - c_i(\alpha_i \cdot m^i_j + r_i))^2 + \alpha_j \cdot m^j_i \cdot \frac{1}{2}(d_j - c_j((\alpha_j \cdot m^j_i + r_j))$$

If $f_i(m_j^i) \equiv \frac{1}{2}(b_i - a_i(\alpha_i \cdot m_j^i + r_i)$ then

$$g(m_a^b, m_b^a) = \sum_{i=a,b} \hat{\pi}_i(m_a^b, m_b^a) = \sum_{i=a,b} \left(\frac{1}{c_i}(f_i(m_j^i))^2 + \alpha_i \cdot m_j^i \cdot f_i(m_j^i) \right)$$

Moreover, by computation

$$\partial g(m_a^b, m_b^a) = \left(-\frac{c_a \cdot \alpha_a^2 \cdot m_b^a}{2}, -\frac{c_b \cdot \alpha_b^2 \cdot m_a^b}{2} \right)$$

It follows that $m_b^a = m_a^b = 0$. In other words one can assert that peering agreements occur in this example.

7.2. SOME OTHER EXTENSIONS

The preceding example opens the way to a large class of applications. In fact, if one comes back to this example, one notices that this one is simply obtained by the manipulation of some data of a transportation game. By taking as given the margins charged by each IBP to the other, one simply considers, from a formal point of view, a new game in which these margins are introduced in the network costs r_i. The cooperative game which is played in the first step therefore simply manipulates the natural parameters of this game.

If one now remembers that the basic parameters are (i) the network costs, (ii) the demands and (iii) the number of agents providing one service, one can think to a large class of other applications which affect these parameters. Moreover, our notion also fits with an one-way network. Thus if one merges these two criteria one obtains a large set of applications which can be summarized in the following table:

	Cost	Demand	Number of agents
Two ways	transportation, congestion	network competition	connectivity, hubs
One way	resource based firms	quality, localisation	vertical integration

For instance, if one considers a two-way network, one can manipulate the unit cost by introducing congestion and therefore enhance our optimal road choice (which is already a cost manipulation problem). The reciprocal access problem can also be revisited under the assumption that there exists network externalities. If this happens the demand for a service is not only related to its price but also to the number of users of a given service. For instance if one chooses a telecommunication company one is also interested in the numbers of friends one is able to join. From that point of view, networks are in competition and these behaviors surely affects the reciprocal access charges. The number of actors in a two-way network is also important. Is there an incentive to open new roads in order to attract more consumers having in mind that new intermediaries try to capture a part of the profit? By answering this question, one directly addresses the problem of the construction of the network, a problem that we escaped in this paper by taking the network as given.

One can even go a step further by noting that our approach also applies to one-way networks. From that point of view, every production process can be viewed as a network because it consists of a set of basic activities which concur to the production of a commodity (Soubayran–Stahn [19]). The production cost of these network firms can be affected by some coordination costs which can be lowered if some efforts are spend. But if these resources are available in limited amount, one deals with a resource based approach of the firm. The demand can also be affected by the quality of the network good or by the place where this good is available. Localization problems in the standard sense as well as in the space of characteristics can therefore be addressed. Finally, if one changes the number of independent decision centers within a production network one addresses the problem of vertical integration (Soubayran–Stahn [18]).

8. Concluding Remarks

In this paper, we studied monopolistic pricing behaviors within a two-way network. Each basic activity which concurs to the production of a network good was identified to an independent decision center which had some market power. Moreover under the assumption that every player knows the network, we have introduced a generalized double marginalization game. This concept depicts a situation in which each actor of the network, due to this peculiar rationality, tries to capture a share of the profit of the firms who sell the system goods to the consumers. Moreover, because we deals with a symbiotic network, i.e. a network in which each final seller also acts as an intermediary, we construct a game in which the margins are chosen simultaneously.

The solution of this game was largely studied in this paper. We showed the existence of such equilibria under rather general conditions on the demand functions. In fact to obtain existence, we only required that the demands are strictly decreasing up to a maximal reservation price. We also introduced a condition which ensures in some sense the uniqueness of the equilibrium. This one relies on the log-concavity of the demand functions and can by based on specific distribution of the reservation prices in a discrete choice model. Under these restrictions, we were able to construct a complete characterization of these equilibria including optimal road choices.

This notion of equilibrium which relies on the fact that the players have a network rationality also opens a wide scope of applications. Some of them were discussed in this paper. They often induce the introduction of a game which precede the capture game and which consists in a manipulation of it basic data that is the network costs, the demand and the number of players. This game can be either cooperative or non-cooperative. In the first case one deals with co-opetition, in the second one with sub-game perfect equilibria. The study of these extensions leads to what we call a more general capture theory which is based on a manipulation of the data.

The model developed in this paper however suffers from several limitations. The reader surely notices that the assumption, commonly used in this literature, which stays that the demand of a network service only depends of his own price largely simplifies the treatment of the model.

It would be interesting to extend this construction in this direction. Moreover, it is well known that uniqueness often requires strong assumptions. But it would be interesting to try to find a weaker assumption than the one of log-concavity of the demand. Finally, it could be interesting to study the peering behaviors in a more general context or even to introduce more general co-opetition problems because these 'games' relies on the choice an optimal solution in a set of parametrized strategic equilibria.

Appendix

A. PROOF OF PROPOSITION 1

At equilibrium, one knows that:

$$((m_a^{*t})_{t\in T_a}) \in \underset{((m_a^{*t})_{t\in T_a})}{\arg\max} \sum_{s\in S(T_a)} \sum_{t\in S^{-1}(s)\cap T_a} \alpha_s^t \cdot m_\alpha^t \cdot D_s\left(\sum_{t\in S^{-1}(s)} \alpha_s^t \left(c(t) + m_a^t + \sum_{\substack{a'\in t\\ a'\neq a}} m^{*t} - a'\right)\right)$$

Hence

$$((m_a^{*t})_{t\in T_a}) \in \underset{((m_a^{*t})_{t\in T_a})}{\arg\max} \sum_{s\in S(T_a)} \left(\sum_{t\in S^{-1}(s)\cap T_a} \alpha_s^t \cdot m_\alpha^t\right) \cdot D_s\left(r_s + \sum_{t\in S^{-1}(s)} \alpha_s^t \cdot m_a^t + \sum_{j\in I_s\setminus\{i\}} \sigma_j^*\right)$$

It remains (i) to notice that this optimization problem is additively separable and (ii) to apply a change of variables in order to conclude that:

$$\forall s \in S(T), \forall i \in I_s, \quad \sigma_{i,s}^* \in \underset{\sigma_{i,s}\in\mathbb{R}_+}{\arg\max}\ \sigma_{i,s} D_s\left(\sigma_{i,s} + \sum_{j\in I_s\setminus\{i\}} \sigma_{j,s}^* + r_s\right) \qquad \square$$

B. PROOF OF PROPOSITION 2

Because $(\tilde{\sigma}_a^s)_{a\in A_s}$ is an equilibrium of the s^{th} generic game it follows that:

$$\forall s \in S(T), \forall a \in A_s, \quad \tilde{\sigma}_a^s \in \arg\max_{\sigma_a^s \in \mathbb{R}_+} \sigma_a^s \cdot D\left(\sigma_a^s + \sum_{a' \in A_s \setminus \{a\}} \tilde{\sigma}_a^s + r_s\right)$$

but $\forall s \in S(T), \forall a \in A_s, \sum_{t\in T_a \cap S^{-1}(s)} \alpha_s^t \cdot \tilde{m}_a^t = \tilde{\sigma}_a^s$, hence:

$$\forall s \in S(T), \forall a \in A_s,$$
$$\left(\sum_{t\in T_a\cap S^{-1}(s)} \alpha_s^t \cdot \tilde{m}_a^t\right) \in \arg\max_{(m_a^t)_{t\in T_a\cap S^{-1}(s)}} \left(\sum_{t\in T_a\cap S^{-1}(s)} \alpha_s^t \cdot m_a^t\right) \cdot D_s(p_s)$$

with $p_s = \sum_{t\in S^{-1}(s)} \alpha_s^t \left(c(t) + \sum_{\substack{a'\in t \\ a'\neq a}} \tilde{m}_{a'}^t\right) + \sum_{t\in T_a\cap S^{-1}(s)} \alpha_s^t \cdot m_a^t$.

From that point of view any vector $(\tilde{m}_a^t)_{t\in T_a\cap S^{-1}(s)}$ of margins which satisfies $\sum_{t\in T_a\cap S^{-1}(s)} \alpha_s^t \cdot \tilde{m}_a^t = \tilde{\sigma}_a^s$ also solves:

$$\forall s \in S(T), \forall a \in A_s,$$
$$(\tilde{m}_a^t)_{t\in T_a\cap S^{-1}(s)} \in \arg\max_{(m_a^t)_{t\in T_a\cap S^{-1}(s)}} \left(\sum_{t\in T_a\cap S^{-1}(s)} \alpha_s^t \cdot m_a^t\right) \cdot D_s(P_s)$$

The reader also notices that for one agent $a \in A_s$ the whole set of margins which appears in the preceding equation for a given $s \in S(T)$ never reappears for a $s' \neq s$. One can therefore sum these optimization problems of $s \in S(T)$ and even claim that $(\tilde{m}_a^t)_{t\in T_a}$ optimizes the sum. Hence:

$$\forall a \in A_s, \quad (\tilde{m}_a^t)_{t\in T_a} \in \arg\max_{(m_a^t)_{t\in T}} \sum_{s\in S(T_a)} \left(\left(\sum_{t\in T_a\cap S^{-1}(s)} \alpha_s^t \cdot m_a^t\right) \cdot D_s(p_s)\right)$$

Finally because each agent $a \in A$ is in at least one A_s, the preceding equation is true $\forall a \in A$. But in this case $((\tilde{m}_a^t)_{t\in T^a})_{a\in A}$ is an equilibrium of the complete game.

C. PROOF OF PROPOSITION 3

Let Σ_{-i}^* be the fixed-point of φ. It follows that $\phi(\Sigma_{-i}^*) = \frac{n}{n-1}\Sigma_{-i}^*$. Moreover we know that:

$$\phi(\Sigma_{-i}^*) \in \arg\max_{\Sigma\in C(\Sigma_{-i}^*)} (\Sigma - \Sigma_{-i}) \cdot D(\Sigma + r)$$

Let us now proceed to a change of variable given by $\sigma_i = \Sigma - \Sigma^*_{-i}$. Having in mind that Σ^*_{-i} is a fixed-point of φ, $\phi(\Sigma^*_{-i})$ becomes $\sigma^*_i = \frac{1}{n-1}\Sigma^*_{-i}$ and one verifies that:

$$\sigma^*_i \in \underset{\sigma_i \in [0, \bar{\Sigma} - \Sigma^*_{-i}]}{\arg\max} \; \sigma_i \cdot D(\sigma + \Sigma^*_{-i} + r)$$

But this means that σ^*_i is for each player the best response to a strategy in which the other players choose $\sigma^*_i = \frac{1}{n-1}\Sigma^*_{-i}$. It is therefore a symmetric Nash equilibrium. Reciprocally, let $\sigma^*_i = \sigma^*$ for $i = 1, \ldots, n$ describe a symmetric Nash equilibrium. It follows that:

$$\sigma^* \in \underset{\sigma_i \in [0, \bar{\Sigma} - (n-1)\sigma^*]}{\arg\max} \; \sigma_i \cdot D(\sigma_i + (n-1)\sigma^* + r)$$

Let $\Sigma^*_{-i} = (n-1)\sigma^*$. By an appropriate translation, the preceding optimization program is equivalent to:

$$\Sigma^* = (n \cdot \sigma^*) \in \underset{\Sigma \in C(\Sigma^*_{-i})}{\arg\max} (\Sigma - \Sigma^*_{-i}) \cdot D(\Sigma + r)$$

and it is a matter of fact to verify that Σ^*_{-i} is fixed-point of φ because $\varphi(\Sigma^*_{-i}) = \frac{n-1}{n}(n \cdot \sigma^*) = (n-1)\sigma^* = \Sigma^*_{-i}$. □

D. PROOF OF PROPOSITION 4

Let us assume the contrary. In this case there exists (σ^*_i) a non-trivial equilibrium, with the property that for at least two agents i_0 and i_1 $\sigma^*_{i_0} \neq \sigma^*_{i_1}$. It follows that $\Sigma^*_{-i_0} = \sum_{i=1, i \neq i_0}^{n} \sigma^*_i \neq \sum_{i=1, i \neq i_1}^{n} \sigma^*_i = \Sigma^*_{-i_1}$. After the usual variable change, one also knows that $\Sigma^* = \sum_{i \in I} \sigma^*_i$ satisfies both:

$$\Sigma^* \in \underset{\Sigma \in C(\Sigma^*_{-i_0})}{\arg\max} (\Sigma - \Sigma^*_{-i_0}) \cdot D(\Sigma + r) \quad \text{and}$$

$$\Sigma^* \in \underset{\Sigma \in C(\Sigma^*_{-i_1})}{\arg\max} (\Sigma - \Sigma^*_{-i_1}) \cdot D(\Sigma + r)$$

As long as $\Sigma^*_{-i_0}, \Sigma^*_{-i_1} < \bar{\Sigma}$, the solution of the two programs are interior ones. This follows from the fact that $\pi(\Sigma^*_{-i}, \Sigma^*_{-i}) = \pi(\bar{\Sigma}, \Sigma^*_{-i}) = 0$ for $i = i_1, i_2$ and that $D(\Sigma + r) > 0$ for $\Sigma < \bar{\Sigma}$ which implies that $\pi\left(\frac{\bar{\Sigma} - \Sigma^*_{-i}}{2}, \Sigma^*_{-i}\right) > 0$ for $i = i_1, i_2$. Hence Σ^* satisfies both first order conditions. They are given by:

$$\begin{cases} (\Sigma^* - \Sigma^*_{-i_0})D'(\Sigma^* + r) + D(\Sigma^* + r) = 0 \\ (\Sigma^* - \Sigma^*_{-i_1})D'(\Sigma^* + r) + D(\Sigma^* + r) = 0 \end{cases}$$

But this implies that $(\Sigma^*_{-i_0} - \Sigma^*_{-i_0})D'(\Sigma^* + r) = 0$ and because $\Sigma^*_{-i_0} \neq \Sigma^*_{-i_1}$ that $D'(\Sigma^* + r) = 0$ which is a contradiction.

In order to be complete, it remains to verify that a non-trivial equilibrium $(\sigma_i^*)_{i\in I}$ has the property that $\Sigma_{-i}^* < \bar{\Sigma}$. So let me assume that $\Sigma_{-i}^* \geqslant \bar{\Sigma}$. Because we consider non-trivial equilibria $\sum_{i\in I}\sigma_i^* \leqslant \bar{\Sigma}$, it follows that $\Sigma_{-i}^* = \bar{\Sigma}$. In this case, any agent $i' \neq i$ has an incentive to lower $\sigma_{i'}$ in order to realize a strictly positive profit. But this is in contradiction with the notion of equilibrium.

E. PROOF OF PROPOSITION 5

It is a matter of facts to verify that $f(\Sigma_{-i})$ is well-defined because $D(p)$ is continuous and $C(\Sigma_{-i}) = [\Sigma_{-i}, \bar{\Sigma}]$ is non empty and compact. Let us now verify that $f(\Sigma_{-i})$ is non decreasing. In order to obtain this result, one first notices that the set-valued function C is monotone nondecreasing because for $\forall \Sigma'_{-i}, \Sigma''_{-i} \in [0, \bar{\Sigma}]$ with $\Sigma'_{-i} < \Sigma''_{-i}$. Concerning this function C, one also observes $\forall \Sigma' \in C(\Sigma'_{-i})$ and $\forall \Sigma'' \in C(\Sigma''_{-i})$, $\min\{\Sigma', \Sigma''\} \in C(\Sigma'_{-i})$ and $\max\{\Sigma', \Sigma''\} \in C(\Sigma''_{-i})$. If one wants to apply theorem 4 (p. 163) of Milgrom–Shannon [16] in order to prove that $f(\Sigma_{-i})$ is non decreasing, it remains to verify that the profit function $\pi(\Sigma, \Sigma_{-i})$ satisfies the strict single crossing property (see Milgrom–Shannon [16]) as long as $D(p)$ is strictly decreasing (that is for non-trivial equilibria). To verify this point let us choose $\Sigma' > \Sigma''$ and $\Sigma'_{-i} > \Sigma''_{-i}$ and let us verify that $\pi(\Sigma', \Sigma''_{-i}) \geqslant \pi(\Sigma'', \Sigma''_{-i}) \Rightarrow \pi(\Sigma'', \Sigma'_{-i})$. By computation, one obtains:

$$
\begin{aligned}
&\pi(\Sigma', \Sigma''_{-i}) \geqslant \pi(\Sigma'', \Sigma''_{-i})\\
&\Leftrightarrow (\Sigma' - \Sigma''_{-i}) \cdot D(\Sigma' + r) \geqslant (\Sigma'' - \Sigma''_{-i}) \cdot D(\Sigma'' + r)\\
&\Leftrightarrow \Sigma' \cdot D(\Sigma' + r) - \Sigma'_{-i} \cdot D(\Sigma' + r)\\
&\geqslant (\Sigma'' - \Sigma''_{-i}) \cdot D(\Sigma'' + r) + (\Sigma''_{-i} - \Sigma'_{-i}) \cdot D(\Sigma' + r)
\end{aligned}
$$

Because $\Sigma''_{-i} - \Sigma'_{-i} < 0$ and $D(\Sigma' + r)$ is strictly decreasing, this implies that:

$$
\begin{aligned}
&\Rightarrow (\Sigma' - \Sigma'_{-i})D(\Sigma' + r) > (\Sigma'' - \Sigma''_{-i})D(\Sigma'' + r) + (\Sigma''_{-i} - \Sigma'_{-i})D(\Sigma'' + r)\\
&\Leftrightarrow (\Sigma' - \Sigma'_{-i})D(\Sigma' + r) > (\Sigma'' - \Sigma'_{-i})D(\Sigma'' + r)\\
&\Leftrightarrow \pi(\Sigma', \Sigma'_{-i}) > \pi(\Sigma'', \Sigma'_{-i})
\end{aligned}
$$

and this ends the proof. □

F. PROOF OF PROPOSITION 6

Obvious.

G. PROOF OF PROPOSITION 7

Let us first check the uniqueness of the fixed point of $\varphi(\Sigma_{-i})$. This property is obtained if $\phi(\Sigma_{-i}) = \{\Sigma \in \mathbb{R}_+ | \Sigma \in \arg\max_{\Sigma \in C(\Sigma_{-i})} \pi(\Sigma, \Sigma_{-i})\}$ is a continuous function with the property $\forall \Sigma_{-i} \in]0, \bar{\Sigma}[$, $\phi'(\Sigma_{-i}) < \frac{n}{n-1}$. In this case $\varphi(\Sigma_{-i}) = \frac{n-1}{n}\phi(\Sigma_i)$ is continuous and $\forall \Sigma_{-i} \in]0, \bar{\Sigma}[$, $\varphi'(\Sigma_{-i}) < 1$. It therefore remains to study the optimization problem which defines $\phi(\Sigma_{-i})$. One first notices that an optimum never meets the boundary of $C(\Sigma_{-i})$ as long as $\Sigma_{-i} < \bar{\Sigma}$. This follows from the fact that $\pi(\Sigma_{-i}, \Sigma_{-i}) = \pi(\bar{\Sigma}, \Sigma_i) = 0$ and $\lim_{\Sigma \to \bar{\Sigma}} \partial_\Sigma \pi(\Sigma, \Sigma_{-i}) < 0$ because $\lim_{p \to \bar{p}^-} D'(p) < 0$. Thus any solution of the preceding program satisfies, by the first order condition, $D(\Sigma + r) + (\Sigma - \Sigma_{-i})D'(\Sigma + r) = 0$. Moreover if $D(p)$ is strictly log-concave on $]0, \bar{p}[$, one verifies that:

$$\forall p \in]0, \bar{p}[, \quad \frac{D''(p) \cdot D(p) - (D'(p))^2}{(D(p))^2} < 0$$

It follows that the second order condition of the preceding optimization problem is also verified. In fact by computation, one obtains:

$$\begin{aligned} &2 \cdot D'(\Sigma + r) + (\Sigma - \Sigma_{-i}) \cdot D''(\Sigma + r) \\ &\quad = 2 \cdot D'(\Sigma + r) - \frac{D(\Sigma + r)}{D'(\Sigma + r)} \cdot D''(\Sigma + r) \\ &\quad = \frac{2 \cdot (D'(\Sigma + r))^2 - D(\Sigma + r) \cdot D''(\Sigma + r)}{D'(\Sigma + r)} < 0 \end{aligned}$$

because the strict log-concavity of $D(p)$ induces that $\forall \Sigma \in]\Sigma_{-i}, \bar{\Sigma}[$

$$\begin{aligned} &2 \cdot (D'(\Sigma + r))^2 - D(\Sigma + r) \cdot D''(\Sigma + r) > (D'(\Sigma + r))^2 \\ &\quad - D(\Sigma + r) \cdot D''(\Sigma + r) > 0 \end{aligned}$$

So let us apply the implicit function theorem to the first order conditions in order to check that $\phi'(\Sigma_{-i}) < \frac{n}{n-1}$. But computation one has:

$$\forall \Sigma_{-i} \in]0, \bar{\Sigma}[\quad \sigma'(\Sigma_{-i}) = \frac{(D'(\Sigma + r))^2}{2 \cdot (D'(\Sigma + r))^2 - D(\Sigma + r) \cdot D''(\Sigma + r)}$$

But log-concavity implies that:

$$\frac{(n+1)}{n} \cdot (D'(\Sigma + r))^2 - D(\Sigma + r) \cdot D''(\Sigma + r) > 0$$

if one multiplies this equation by n and adds $(n-1) \cdot (D'(\Sigma + r))^2$ to the two members one obtains:

$$(2n \cdot (D'(\Sigma + r))^2 - n \cdot D(\Sigma + r) \cdot D''(\Sigma + r)) > (n-1) \cdot (D'(\Sigma + r))^2$$

Hence $\phi'(\Sigma_{-i}) < \frac{n}{n-1}$.

Let us now compute the profit of a firm at an equilibrium. Because an equilibrium is deduced from a fixed point of $\varphi(\Sigma_{-i})$ this one also satisfies $\phi(\Sigma_i)$, one even knows that $\phi^{-1}(\Sigma) = \frac{D(\Sigma+r)}{D'(\Sigma+r)} + \Sigma$. Thus $\Sigma \equiv \frac{n}{n-1}\Sigma_{-i}$ verifies $-\frac{D(\Sigma+r)}{D'(\Sigma+r)} =$

$\frac{1}{n}\Sigma \equiv \sigma_i$. The profit of a firm is therefore, at equilibrium, given by $\pi(\Sigma, r) = -\frac{D(\Sigma+r)}{D'(\Sigma+r)} \cdot D(\Sigma + r)$ with $-\frac{D(\Sigma+r)}{D'(\Sigma+r)} = \frac{1}{n}\Sigma$. In order to simplify the computation of the derivatives of this function, let us now introduce a variable change given by $p = \Sigma + r$ and let us define $h(p) = \frac{D(p)}{-D(p)} > 0$. The profit becomes $\pi(p) = D(p) \cdot h(p)$ with $\frac{p-r}{n} = h(p)$. Because $h(p) = -\left(\frac{d(\log(D(p)}{dp}\right)^{-1}$, it follows by the log-concavity assumption that $h'(p) < 0$. By the implicit function theorem, one also remarks that $\frac{\partial p}{\partial n} = \frac{h(p)}{1-nh'(p)} > 0$ and that $\frac{\partial p_F}{\partial r_F} = \frac{1}{1-nh'(p)} > 0$. It remains to notice that $\frac{d\pi}{dp} = D'(p) \cdot h(p) + h'(p) \cdot D(p) < 0$ in order to conclude that $\partial_n \pi(n, r) < 0$ and that $\partial_r \pi(n, r) < 0$. □

H. PROOF OF PROPOSITION 8

Let us consider the following optimization program

$$\max_{((\alpha_s^t)_{t\in S^{-1}(s)})} \pi_s(\#A_s, r_s) \quad \text{s.t.} \quad \sum_{t\in S^{-1}(s)} \alpha_s^t = 1$$

One first notices that a firm only activates one travel. Let us suppose the contrary and let us denote by t_1 and t_2 two activated travels. Because the travels are different the number of activities $\#A_S$ involved in the production of the service s satisfies $\#A_S > \max\{n(t_1), n(t_2)\}$. Moreover $r_s = \alpha \cdot c(t_1) + (1-\alpha) \cdot c(t_2) > \min\{c(t_1), c(t_2)\}$ for $\alpha \in]0, 1[$. By Proposition 7, it follows that:

$$\forall \alpha \in]0, 1[\quad \pi_s(\#A_s, r_s) < \pi_s(\max\{n(t_1), n(t_2)\}, \min\{c(t_1), c(t_2)\})$$

Finally if for instance $c(t_1) = \min\{c(t_1), c(t_2)\}$ then $\pi_s(n(t_1), c(t_1))$ which is the desired contradiction.

The optimal road choice program can therefore be rewritten as:

$$\max_{t\in S^{-1}(s)} \{\pi_s(n(t), c(t))\}$$

But one also knows by assumption that shorter travels are cheaper (i.e. $n(t) < n(t') \Rightarrow c(t) < c(t')$). This firm has therefore a strong incentive to choose the travel $t \in T_s^n = \left\{t \in T | t \in \arg\min_{t\in S^{-1}(s)}\{n(t)\}\right\}$. Moreover because travels of the same length may not have the same cost. One can restrict the set of optimal travels to $T_s^* = \left\{t \in T | t \in \arg\min_{t\in T_n}\{c(t)\}\right\}$. The optimal strategy therefore consists in randomly choosing a travel in the set of cheapest and shortest travel T_s^*.

I. PROOF OF PROPOSITION 9

Obvious.

References

1. Amir, R. (1996), Cournot Oligoploy and the Theory of Supermodular Games, *Games and Economic Behaviors* 15: 132–148.
2. Caplin, A. and Nalebuff, B. (1991), Aggregation and Imperfect Competition: on the Existence of Equilibrium, *Econometrica* 59: 25–59.
3. Carter, M. and Wright, J. (1994), Symbiotic Production: the Case of Telecommunication Pricing, *Review of Industrial Organization* 9: 365–378.
4. Crémer, J., Rey, P. and Tirole, J. (1999), Connectivity in Commercial Internet. *Working paper IDEI*.
5. Cricelli, L., Gastaldi, M. and Levialdi, N. (1999), Vertical integration in International Telecommunication System, *Review of Industrial Organization* 14: 337–353.
6. Dang Nguyen, G. and Pénard, T. (1999), *Les accords d'interconnexion dans internet: enjeux économiques et perspectives réglementaires, Département d'économie*, ENST, Bretagne.
7. Economides, N. and White, L.J. (1994), Networks and Compatibility: Implications for Antitrust, *European Economic Review* 38: 651–662.
8. Hendricks, K., Piccione, M. and Tan, G. (1995), The Economics of Hubs: The case of Monopoly, *Review of Economic Studies* 62: 83–99.
9. Hendricks, K. Piccione, M. and Tan, G. (1999), Equilibria in networks, *Econometrica* 67: 1407–1437.
10. Laffont, J-J. Rey, P. and Tirole, J. (1997), Network Competition I: Overview and Nondiscriminatory Pricing, *Rand Journal of Economics* 29: 1–37.
11. Laffont, J-J., Rey, P. and Tirole, J. (1997), Network Competition II: Overview and Nondiscriminatory Pricing, *Rand Journal of Economics* 29: 38–56.
12. Laffont, J-J. and Tirole, J. (1999), *Competition in Telecommunications*, MIT Press.
13. Long, N.V. and Soubeyran, A. (1999), *Cost Manipulation Games in Oligopoly with Cost of Manipulating*, CIRANO WP 99-13, Montréal.
14. Matutes, C. and Regibeau, P. (1988), Mix and Match: Product Compatibility without Network Externalities, *Rand Journal of Economics* 19: 221–234.
15. Milgrom, P. and Shannon, C. (1994), Monotone Comparative Statics, *Econometrica* 62: 157–180.
16. Nalebuff, B.J. and Brandenburger, A.M. (1996), *Co-opetition*, Doubleday, Bantam Doubleday Dell Publishing Group.
17. Soubeyran, A. and Stahn, H. (2000), *Profit Capture and Optimal Size of a Network Firm*. Mimeo.
18. Soubeyran, A. and Stahn, H. (2001), *The firm as a manipulating organization of its network capure game.* Mimeo.
19. Tarski, A. (1955), A Lattice-Theoretical Fixpoint Theorem and its Applications, *Pacific Journal of Mathematics* 5: 285–309.